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A PROMISING COUPLING OF DAFTARDAR-JAFARI METHOD AND HE'S FRACTIONAL DERIVATION TO APPROXIMATE SOLITARY WAVE SOLUTION OF NONLINEAR FRACTIONAL KDV EQUATION

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Abstract. In this article, fractional Korteweg-de Vries equation (fKdVe) is considered in a fractal domain. The fractional complex transform and He's fractional derivative are applied for converting the mentioned fractional equation into an ordinary differential equation. The Daftardar-Jafari iteration approach is investigated to analyse the suggested model. The convergence, stability and error analysis are discussed. Meanwhile, the thrust of the present paper is investigated by several examples.

Keywords: Fractional complex transform, Daftardar-Jafari method, He's fractional derivative, Korteweg-de Vries equation.

AMS Subject Classification: 74J35, 76B25, 35Q53, 35R11.

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1 Introduction

The fKdVe is derived from the propagation of dispersive shallow water waves. This applicable equation is used in continuum mechanics, fluid dynamics, solitons turbulence, aerodynamics, mass transport and boundary layer behavior. For more details see Jafari et al. (2008) and references therein.

In this paper, the following fKdVe is considered (Jafari et al., 2008)

$$\frac{\partial^\alpha}{\partial t^\alpha}\eta(x,t) + a\eta^p(x,t)\frac{\partial^\alpha}{\partial x^\alpha}\eta(x,t) + b\frac{\partial^{3\alpha}}{\partial x^{3\alpha}}\eta(x,t) = 0, \tag{1}$$

where $a, b \neq 0$ are arbitrary constants coefficients and not equal to zero, $\eta(x,t)$ is a field variable, and $t \in T(= [0, t_0](t_0 > 0))$ is the time. Furthermore, $\frac{\partial^\alpha}{\partial t^\alpha}$ is He's fractional derivative defined in the following form (He, 2014; Hi & Sun, 2019; Liu et al., 2014; Li & He, 2020; Wang & Wang, 2019; Wang & He, 2019)

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_{t_0}^{t}(s-t)^{n-\alpha-1}[\eta_0(s) - \eta(s)]ds. \tag{2}$$

Here with the same initial condition, η_0 is the solution of its continuous partner of the equation. For more details regarding the properties of fractional calculus and their applications, the reader is advised to consult the results of the research works presented in Benson et al. (2013); Gorenflo et al. (2001); Hilfert (2000); Lundstrom et al. (2008); Meerschaert et al. (1999); Metzler & Klafter (1997); Rossikhin & Shitikova (1997); Roop (20063); Sabatelli et al. (2002); Schumer et al. (2001, 2003).

1.1 Fractional complex transform

In the follow up we consider the following general differential equation of fractional order

$$f(\eta, \eta_t^{\alpha}, \eta_x^{\beta}, \eta_t^{2\alpha}, \eta_x^{2\beta}, \cdots) = 0,\ 0 < \alpha \leq 1,\ 0 < \beta \leq 1. \tag{3}$$

To illustrate the fractional complex scheme Su et al. (2013), assume know that

$$\tau = \frac{c_1 t^{\alpha}}{\Gamma(1+\alpha)}, \text{ and } \quad \varsigma = \frac{c_2 x^{\beta}}{\Gamma(1+\beta)}, \tag{4}$$

where c_1, c_2 are arbitrary unknown constants. Using the essential properties of fractional calulus we have

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} = c_1 \frac{\partial \eta}{\partial \tau}, \text{ and } \quad \frac{\partial^{\beta}\eta}{\partial t^{\beta}} = c_2 \frac{\partial \eta}{\partial \tau}, \tag{5}$$

accordingly, the fractional differential equations can be converted into ordinary differential equations.

1.2 The Daftardar-Jafari method

Consider the following equation

$$\eta(\overline{x}) - N(\eta(\overline{x})) = f(\overline{x}),\ \overline{x} = (x_1, x_2, \cdots, x_n), \tag{6}$$

where f is an arbitrary known function and N is a nonlinear operator. The solution of Eq. (6) has the following series form Daftardar-Gejji & Jafari (2006)

$$\eta(\overline{x}) = \sum_{i=0}^{\infty} \eta_i(\overline{x}). \tag{7}$$

The operator N can be written as follows

$$N(\sum_{i=0}^{\infty} \eta_i) = N(\eta_0) + \sum_{i=1}^{\infty} \left\{ N(\sum_{j=0}^{i} \eta_i) - N(\sum_{j=0}^{i-1} \eta_j) \right\}. \tag{8}$$

According to Eqs. (6) and (7), Eq. (8) can be written as

$$\sum_{i=0}^{\infty} \eta_i = f + N(\eta_0) + \sum_{i=1}^{\infty} \left\{ N(\sum_{j=0}^{i} \eta_i) - N(\sum_{j=0}^{i-1} \eta_j) \right\}. \tag{9}$$

In the follow up, one will set

$$\begin{aligned} \eta_0 &= f, \\ \eta_1 &= N(\eta_0), \\ \eta_{m+1} &= N(\eta_0 + \eta_1 + \cdots + \eta_m) - N(\eta_0 + \eta_1 + \cdots + \eta_{m-1}),\ m = 1, 2, 3, \cdots. \end{aligned} \tag{10}$$

Consequentely

$$(\eta_1 + \eta_2 + \cdots + \eta_{m+1}) = N(\eta_0 + \eta_1 + \cdots + \eta_m),\ m = 1, 2, 3, \cdots, \tag{11}$$

and

$$\sum_{i=0}^{\infty} \eta_i - N(\sum_{i=0}^{\infty} \eta_i) = f. \tag{12}$$

The q-term approximate solution of Eq. (6) can be given as follows

$$\eta = \eta_0 + \eta_1 + \cdots + \eta_{q-1}. \tag{13}$$

If N is a contraction, i.e, $|N(x) - N(y)|| \leq q||x-y||, 0 < q < 1$, then

$$||\eta_{m+1}|| = ||N(\eta_0+\eta_1+\cdots+\eta_m)-N(\eta_0+\eta_1+\cdots+\eta_{m-1})|| \leq q||\eta_m|| \leq q^m||\eta_0||,\ m = 0,1,2,3,\cdots. \tag{14}$$

In other words, in view of the Banach fixed point theorem Cherruault (1988) the series $\sum_{i=0}^{\infty} \eta_i$ absolutely and uniformly converges to a solution of Eq. (6).

2 Implementation and applications

Assume know that

$$\tau = \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad \text{and} \quad \varsigma = \frac{x^\beta}{\Gamma(1+\beta)}. \tag{15}$$

Eq. (1) can be converted into a differential equation, as follows

$$\frac{\partial}{\partial \tau}\eta(\varsigma,\tau) + a\eta^p(\varsigma,\tau)\frac{\partial}{\partial \varsigma}\eta(\varsigma,\tau) + b\frac{\partial^3}{\partial \varsigma^3}\eta(\varsigma,\tau) = 0. \tag{16}$$

The zero order solitary wave solution can be taken as the initial value of the state variable, as follows

$$\eta_0(x,t) = \eta(x,0) = k\text{sech}^{\frac{2}{p}}\left(\frac{p}{2\sqrt{b}}\left(\frac{x^\alpha}{\Gamma(\alpha+1)} + \eta_0\right)\right), \tag{17}$$

where $k = \left(\frac{(p+1)(p+2)}{2a}\right)^{\frac{1}{p}}$ and η_0 is an arbitrary constant. Using the Daftardar-Jafari method, the following results can be obtained

$$\begin{cases} \eta(\xi,\tau) = \eta_0(\xi,\tau) + I_T\left(-a\eta^p(\varsigma,\tau)\frac{\partial}{\partial\varsigma}\eta(\varsigma,\tau) - b\frac{\partial^3}{\partial\varsigma^3}\eta(\varsigma,\tau)\right), \\ N(\eta) = I_T\left(-a\eta^p(\varsigma,\tau)\frac{\partial}{\partial\varsigma}\eta(\varsigma,\tau) - b\frac{\partial^3}{\partial\varsigma^3}\eta(\varsigma,\tau)\right), \end{cases} \tag{18}$$

where I is an integral operator. According to Eq. (10), the iteration solution of Eq. (16) can be written as follows

$$\begin{cases} \eta_0(\xi,\tau) = k\text{sech}^{\frac{2}{p}}\left(\frac{p}{2\sqrt{b}}(\xi+\eta_0)\right), \\ \eta_1(\xi,\tau) = k\text{sech}^{\frac{2}{p}}\left(\frac{p}{2\sqrt{b}}(\xi+\eta_0)\right)\frac{t}{\sqrt{b}}\tanh\left(\frac{p}{2\sqrt{b}}(\xi+\eta_0)\right), \end{cases} \tag{19}$$

and

$$\begin{aligned}
&\eta_2(\xi,\tau) = t\Big[\tfrac{ak^{p+1}}{\sqrt{b}}\ \text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tanh(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) \\
&+\tfrac{k}{\sqrt{b}}\text{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tanh^3(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) - \tfrac{3kp}{2\sqrt{b}}\text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) \\
&-\tfrac{kp^2}{2\sqrt{b}}\text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tanh(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) - k\text{sech}^{2/p}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tfrac{t}{\sqrt{b}}\tanh(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) \\
&-\tfrac{kp^2}{2\sqrt{b}}\text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tanh(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) - k\text{sech}^{2/p}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tfrac{1}{\sqrt{b}}\tanh(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\Big] \\
&-t^2\Big[\tfrac{ak^{p+1}}{\sqrt{b}}\ \text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\left(\tfrac{p}{2} - \tfrac{3p+2}{2}\tanh^2(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\right) \\
&-\tfrac{3kp^2(p+2)}{8b}\text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) + \tfrac{3kp(p+2)^2}{8b}\text{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) \\
&\tanh^2(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) + \tfrac{kp^3}{8b}\text{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) - \tfrac{k(p+2)^3}{8b}\text{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tanh^4(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) \\
&+\tfrac{3k(p+2)^2}{8b}\text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tanh^2(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) + \tfrac{kp^2(p+2)}{4b}\text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tanh^2(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\Big] \\
&-t^3\Big[\tfrac{apk^{p+1}}{2b\sqrt{b}}\ \text{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\tanh(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\left(p - \tfrac{p+2}{2}\right)\tanh^2(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0))\Big].
\end{aligned} \tag{20}$$

Hence, we can write the 3th-order approximate solution (ψ_3) of Eq. (16) as the follows

$$\begin{aligned}
&\psi_3 = \eta_0(x,t) + \eta_1(x,t) + \eta_2(x,t) \\
&= k\mathrm{sech}^{2/p}(\tfrac{p}{2\sqrt{b}}(\xi+\eta_0)) + \tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big[\tfrac{ak^{p+1}}{\sqrt{b}}\ \mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&+\tfrac{k}{\sqrt{b}}\mathrm{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh^3(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) - \tfrac{3kp}{2\sqrt{b}}\mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&-\tfrac{kp^2}{2\sqrt{b}}\mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&-k\mathrm{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(+\tfrac{x^\alpha}{\Gamma(\alpha+1)}\eta_0))\tfrac{t^\alpha}{\Gamma(\alpha+1)\sqrt{b}}\tanh(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&-\tfrac{kp^2}{2\sqrt{b}}\mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\Big] \\
&-\Big(\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big)^2\Big[\tfrac{ak^{p+1}}{\sqrt{b}}\ \mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\Big(\tfrac{p}{2}-\tfrac{3p+2}{2}\tanh^2(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\Big) \\
&-t^2\Big[\tfrac{ak^{p+1}}{\sqrt{b}}\ \mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\Big(\tfrac{p}{2}-\tfrac{3p+2}{2}\tanh^2(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\Big) \\
&-\tfrac{3kp^2(p+2)}{8b}\mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) + \tfrac{3kp(p+2)^2}{8b}\mathrm{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&\tanh^2(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) + \tfrac{kp^3}{8b}\mathrm{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&-\tfrac{k(p+2)^3}{8b}\mathrm{sech}^{\frac{2}{p}}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh^4(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&+\tfrac{3k(p+2)^2}{8b}\mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh^2(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&+\tfrac{kp^2(p+2)}{4b}\mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh^2(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0)) \\
&+\tfrac{kp^2(p+2)}{4b}\mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh^2(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\Big] \\
&-\Big(\tfrac{t^\alpha}{\Gamma(\alpha+1)}\Big)^3\Big[\tfrac{apk^{p+1}}{2b\sqrt{b}}\ \mathrm{sech}^{\frac{2}{p}+2}(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\tanh(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\big(p-\tfrac{p+2}{2}\big)\tanh^2(\tfrac{p}{2\sqrt{b}}(\tfrac{x^\alpha}{\Gamma(\alpha+1)}+\eta_0))\Big].
\end{aligned} \tag{21}$$

Here under, if m tends to infinity, then the iteration leads to the solitary wave solution of the fKdVe

$$\eta(x,t) = k\mathrm{sech}^{\frac{2}{p}}(\frac{p}{2\sqrt{b}}(\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)} + \eta_0)). \tag{22}$$

Theorem 1. *Suppose that the Gejji-Jafari iteration method satisfies the conditions (20) and (21). Furthermore, suppose that*

$$F(u) = -a\eta^p(x,t)\frac{\partial}{\partial x}\eta(x,t) - b\frac{\partial^3}{\partial x^3}\eta(x,t), \tag{23}$$

is a continuous function and satisfies a Lipschitz condition on its arguments. Then, our proposed scheme to approximate the solution of Eq. (1) is convergent.

Proof. As before, we define the recurrence relation

$$\begin{cases} \eta_0 = \eta(x,t), \\ \eta_1 = I_T(-a\eta_0^p(x,t)\dfrac{\partial}{\partial x}\eta_0(x,t) - b\dfrac{\partial^3}{\partial x^3}\eta_0(x,t)),\ m = 1,2,3,\cdots \\ \eta_{m+1} = N(\eta_0+\eta_1+\ldots+\eta_m) - N(\eta_0+\eta_1+\ldots+\eta_{m-1}), \end{cases} \tag{24}$$

we can rewrite η_1 in terms of the kernel $K(x,t)$

$$\eta_1 = I_T K(x,t)\eta_0(x,t). \tag{25}$$

Assume now that

$$K_1 = \|K(x,t)\|_\infty < \infty,\ K_2 = \|\eta_0(x,t)\|_\infty < \infty. \tag{26}$$

Prove that $\sum_{i=0}^{\infty}\eta_i$ is uniformly convergent.

$$\begin{cases} |\eta_1(x,t)| \le \int_0^{\bar{t}} |K(x,t)\eta_0(x,t)|dt \le K_1K_2\delta, \\ |\eta_2(x,t)| = |N(\eta_0(x,t)+\eta_1(x,t)) - N(\eta_0(x,t))| \le L\int_0^{\bar{t}}|\eta_1(x,t)|dt \le LK_1K_2\dfrac{\delta^2}{2!}, \\ \vdots \\ |\eta_{m+1}| = |N(\eta_0+\ldots+\eta_m) - N(\eta_0+\ldots+\eta_{m-1}))| \le L\int_0^{\bar{t}}|\eta_m(x,t)|dt \le LK_1K_2\dfrac{\delta^{(m+1)}}{(m+1)!}, \end{cases} \tag{27}$$

where $|\bar{t}| \le \delta$. Hence $\sum_{i=0}^{\infty} \eta_i$ is absolutely and $\eta(x,t)$ uniformly convergent and satisfies Eq. (18).

$$\frac{\partial}{\partial t}\mu(x,t) + a\mu^p(x,t)\frac{\partial}{\partial x}\mu(x,t) + b\frac{\partial^3}{\partial x^3}\mu(x,t) = 0, \tag{28}$$

where

$$|\eta(x,0) - \mu(x,0)| \le \varepsilon. \tag{29}$$

Hence

$$\begin{cases} |\eta_0(x,t) - \mu_0(x,t)| \le \varepsilon, \\ |\eta_1(x,t) - \mu_1(x,t)| \le \int_0^{\bar{t}} K(x,t)|\eta_0(x,t) - \mu_0(x,t)| \le \varepsilon K_1\delta, \\ |\eta_2(x,t) - \mu_2(x,t)| \le L\int_0^{\bar{t}} |\eta_1(x,t) - \mu_1(x,t)|dt \le \varepsilon L K_1 \dfrac{\delta^2}{2!}, \\ \vdots \\ |\eta_{m+1}(x,t) - \mu_{m+1}(x,t)| \le L\int_0^{\bar{t}} |\eta_m(x,t) - \mu_m(x,t)|dt \le \varepsilon L K_1 \dfrac{\delta^{(m+1)}}{(m+1)!}. \end{cases} \tag{30}$$

Hence

$$|\sum_{i=0}^{m+1} \eta_i(x,t) - \sum_{i=0}^{m+1} \mu_i(x,t)| \le \varepsilon\left(1 + K_1\delta + LK_1\frac{\delta^2}{2!} + \ldots + LK_1\frac{\delta^{(m+1)}}{(m+1)!}\right), \tag{31}$$

and this completes the proof. It concludes from this theorem that small changes in initial conditions cause only small changes of the obtained solution. □

Theorem 2. *Under the assumptions of Theorem (1), the Eq. (18) has a solution which is stable.*

Proof. Consider the fKdVe in the form (18). Suppose that and be the solutions of Eq. (18) and the following equation, respectively

$$\frac{\partial}{\partial t}\mu(x,t) + a\mu^p(x,t)\frac{\partial}{\partial x}\mu(x,t) + b\frac{\partial^3}{\partial x^3}\mu(x,t) = 0, \tag{32}$$

where

$$|\eta(x,0) - \mu(x,0)| \le \varepsilon. \tag{33}$$

Hence

$$\begin{cases} |\eta_0(x,t) - \mu_0(x,t)| \le \varepsilon, \\ |\eta_1(x,t) - \mu_1(x,t)| \le \int_0^{\bar{t}} K(x,t)|\eta_0(x,t) - \mu_0(x,t)|dt \le \varepsilon K_1\delta, \\ |\eta_2(x,t) - \mu_2(x,t)| \le L\int_0^{\bar{t}} |\eta_1(x,t) - \mu_1(x,t)|dt \le \varepsilon L K_1 \dfrac{\delta^2}{2!}, \\ \vdots \\ |\eta_{m+1}(x,t) - \mu_{m+1}(x,t)| \le L\int_0^{\bar{t}} |\eta_m(x,t) - \mu_m(x,t)|dt \le \varepsilon L K_1 \dfrac{\delta^{(m+1)}}{(m+1)!}. \end{cases} \tag{34}$$

Hence

$$|\sum_{i=0}^{m+1} \eta_i(x,t) - \sum_{i=0}^{m+1} \mu_i(x,t)| \le \varepsilon\left(1 + K_1\delta + LK_1\frac{\delta^2}{2!} + \ldots + LK_1\frac{\delta^{(m+1)}}{(m+1)!}\right), \tag{35}$$

and this completes the proof. It concludes from this theorem that small changes in initial conditions cause only small changes of the obtained solution. □

Theorem 3. *Under the assumptions of Theorem (1), the upper bound of error is*

$$e_m = |\eta(x,t) - \sum_{i=0}^{m} \eta_i(x,t)| \le LK_1K_2\frac{\theta^{(m+1)\alpha}}{\Gamma(1+(m+1)\alpha)}, \tag{36}$$

where $|\bar{t}| \le \theta$.

Proof. We prove this theorem by induction. For $m = 1$, we have

$$e_1 = |\sum_{i=0}^{2} \eta_i(x,t) - \sum_{i=0}^{1} \eta_i(x,t)| \leq LK_1K_2\frac{\delta^2}{2!}. \quad (37)$$

Assume now that, (35) holds for $m-1$, it immediately follows from (32) that

$$e_m = |\sum_{i=0}^{m+1} \eta_i(x,t) - \sum_{i=0}^{m} \eta_i(x,t)| \leq LK_1K_2\frac{\delta^{(m+1)}}{(m+1)!}. \quad (38)$$

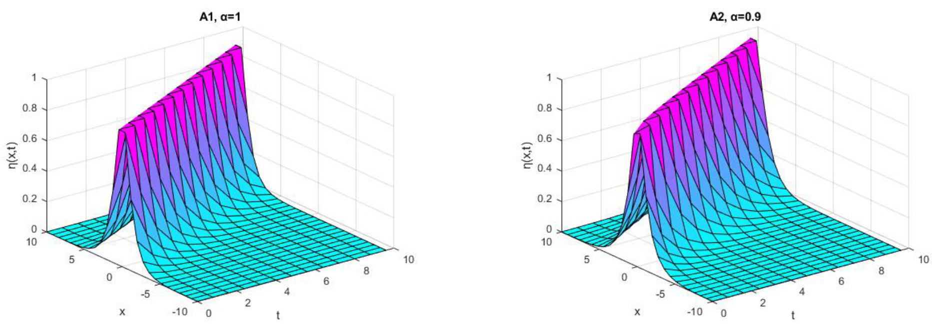
But, from Theorem 1, $\eta(x,t) = \lim_{m+1\to\infty} \sum_{i=0}^{m+1} \eta_i(x,t)$, and therefore

$$e_m = |\eta(x,t) - \sum_{i=0}^{m} \eta_i(x,t)| \leq LK_1K_2\frac{\delta^{(m+1)}}{(m+1)!}. \quad (39)$$

□

3 Discussion

In this study for different meaningful values of $\alpha = 0.7, 0.8, 0.9, 1.0$ $b = 1, \eta_0 = 0, p = 3$ and $a = 10$, the solitary wave solution of fKdVe is obtained. In Fig.1 (A1,A2,A3 and A4) the approximate solutions are demonstrated where the solution η is still a single soliton wave solution for different values of α and the balancing scenario between nonlinearity and dispersion is still valid. Fig.2 (B1 and B2) present the change of amplitude and width of the soliton due to the variation of the order α. The obtained outputs show that the different values of α are uniform both the height and the width of the solitary wave solution. In other words, the order of fractional derivative can be applied for modifying the shape of the solitary wave without any change of the dispersion effects and non-linearity in the medium. In Fig.3 (C1 and C2) at different time values, the behavior between the fractional order and amplitude of the soliton is explained. Results demonstrated that at the same time, the increasing of fractional order increases the amplitude of the solitary wave to some values of α.



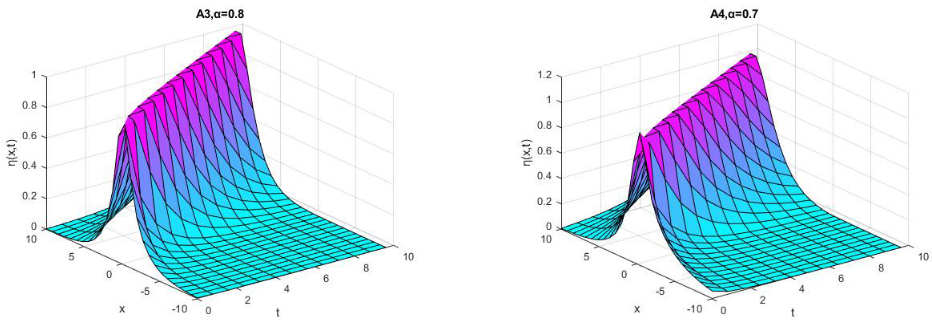


Figure 1: The approximate solutions of Eq. (1) for $\alpha = 1, 0.9, 0.8, 0.7$ are plotted in A1, A2, A3 and A4.

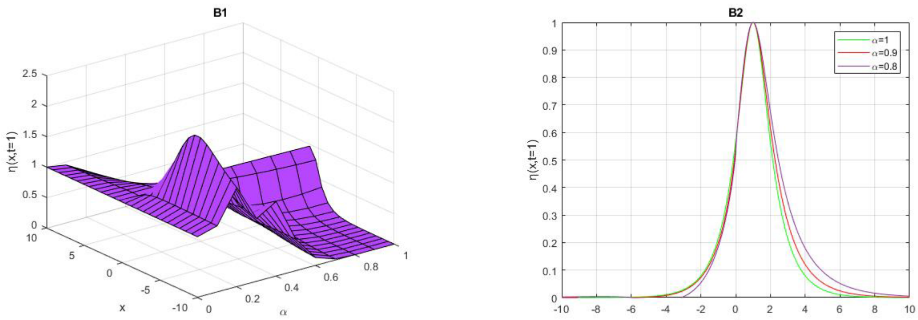


Figure 2: The change of amplitude and width of the soliton due to the variation of the order $\alpha \in [0, 1]$(B1). Also, the change of amplitude and width of the soliton due to the variation of the order $\alpha = 0.8, 0.9, 1$ (B2).

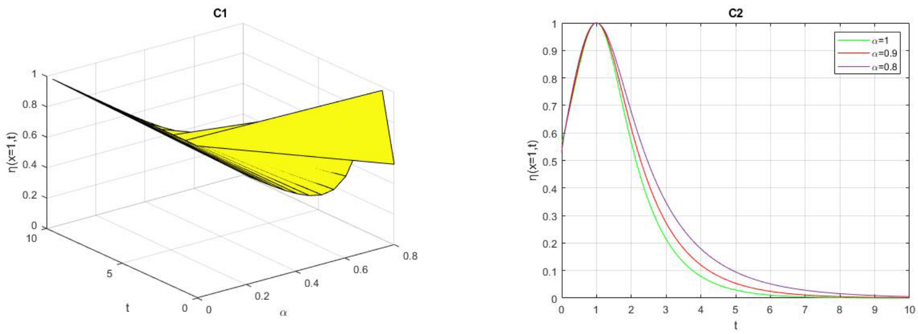


Figure 3: The behavior between the fractional order and amplitude of the soliton of Eq. (1) when $\alpha \in [0, 1]$ and $t \in [0, 10]$ (C1). The behavior between the fractional order and amplitude of the soliton of Eq. (1) when $\alpha = 0.8, 0.9, 1$ and $x = 1$ (C2).

4 Conclusion

In this article, the Daftardar-Jafari method and fractional complex transform thoroughly investigated for finding the approximate solution of the nonlinear fKdVe, where the fractional operator is defined based on He's derivation. The obtained output proves that the proposed scheme is an efficient, robust and simple toll to solve the nonlinear fractional differential equations in the fields of sciences and engineering. Also, we pointed out that the corresponding analytical and numerical solutions were obtained on an Intel CORE i7 laptop by means of some programming codes written in *MATLAB* software.

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